Gravity Train Solution

Let $x(t)$ be the coordinate along the tunnel with the origin placed in the middle, r the distance from the train to the center of the Earth. Then

$$x = \pm r \cos \alpha, \tag{1}$$

where α is the angle between the direction to the middle of the tunnel, and vertical direction.

The gravity force acting on the train is equal to

$$F = -\gamma m M(r) r^{-2},$$

where m is the mass of the train, $M(r)$ the mass of the part of the Earth *below* the train, and γ the gravitational constant. (This expression of the force follows from two theorems of Newton, mentioned in the hints; the mass of the part of the Earth above the train has no influence). Now we have $M(r) = (4\pi/3)\rho r^3$, where ρ is the density of the Earth. Substituting this expression for $M(r)$, we obtain

$$F = -\frac{4}{3}\pi\rho\gamma m r.$$

We only need the component of this force, acting *along the tunnel*; the rest is compensated by the rail reaction. The component along the tunnel is $F \cos \alpha$, where α means the same as in (1).

Thus the equation of motion (Newton's Second Law) and (1) give

$$m x'' = F \cos \alpha = -\frac{4}{3}\pi\rho\gamma m r \cos \alpha = -\frac{4}{3}\pi\rho\gamma m x,$$

where we have chosen the minus sign from the evident physical considerations. Thus our train behaves like a linear oscillator

$$x'' = -k^2 x, \quad \text{where} \quad k^2 = \frac{4}{3}\pi\rho\gamma.$$

As we know, the solutions are periodic with half-period π/k. Half-period is of course the time of a one way trip. Surprisingly, this time does not depend on the length of the tunnel. This means that a trip from West Lafayette to Chicago has the same length as a trip to Paris, or to New Zealand! This means that in addition to fuel efficiency and saving the environment, we also have substantial savings on printing the schedules:-)

Now let us find the length of a one way trip. It is

$$\frac{\pi}{k} = \frac{\pi}{\sqrt{(4/3)\pi\rho\gamma}}.$$

Let us stop for a moment and think how amazing this formula is. It shows that the time of travel depends only on density and the gravitational constant. This means that on all planets made of the same material as the Earth, the time will be the same!

I explained once how to calculate the denominator without going to a library: the acceleration of gravity on the Earth surface is $g = (4/3)\pi\rho\gamma R$, where R is the radius of Earth, so $(4/3)\pi\rho\gamma = g/R = 9.8 \times (\pi/2) \times 10^{-7}$, because the circumference of Earth is 40 millions meters. Now we take a calculator and come with the following answer: 2532 sec=42 min 12 sec. Not very long, even for a trip to Chicago, not speaking of New Zealand.

The maximal speed of the train is also of some interest. It is reached in the middle, when $r = r_0$, and can be found from the Energy Conservation Law:

$$v_{\max}^2 = \frac{4}{3}\pi\rho\gamma(R^2 - r_0^2).$$

Thus

$$v_{\max} = \sqrt{gR - \frac{gr_0^2}{R}}.$$

The largest speed will be reached of course by a train passing through the center of the Earth, when $r = 0$, and $v_{\max} = 7900$ m/sec. This is the so called 1-st cosmic speed (why?!).

Remarks Thanks to Graham Light who corrected an error in the above solution.

There is some evidence that these calculations were essentially known to Hooke, who explained in a letter to Newton, how a body will move inside the Earth if there is no resistance (Newton's first guess about this was wrong, though much later he claimed that he knew the Law of Gravity long before the time of this correspondence. In XIX century a gravity Train Project was seriously offered to Paris Academy of Sciences. Early in XX century this was a popular topic in elementary mechanics textbooks. I learned the idea from one such textbook, which I read in the middle 1960-s. In 1966 one US physicist (Paul Cooper) rediscovered the Gravity Train and published a paper in American Journal of Physics as a serious project of transportation of the future. This was noticed by a journalist from the Time magazine, who published an article (Time, 11 Feb., 1966, p. 42) representing the idea as a science news. A heated discussion in American Journal of Physics followed, but nobody mentioned that the idea is in fact about 400 years old.